NEWS
OF THENATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X **https://doi.org/10.32014/2020.2518-1726.1**
Volume 1, Number 329 (2020), 5 – 13

UDK 517.956
IRSTI 27.31.17

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RESEARCH OF MULTIPERIODIC SOLUTIONS OF PERTURBED LINEAR AUTONOMOUS SYSTEMS WITH DIFFERENTIATION OPERATOR ON THE VECTOR FIELD

Abstract. A linear system with a differentiation operator *D* in the directions of vector fields of the form of the Lyapunov's system with respect to space independent variables and a multiperiodic toroidal form with respect to time variables is considered. All input data of the system multiperiodic depend on time variables or don't depend on them. In this case, some input data received perturbations depending on time variables. We study the question of representing the required motion described by the system in the form of a superposition of individual periodic motions of rationally incommensurable frequencies. The initial problems and the problems of multiperiodicity of motions are studied. It is known that when determining solutions to problems, the system integrates along the characteristics outgoing from the initial points, and then, the initial data are replaced by the first integrals of characteristic systems. Thus, the required solution consists of the following components: characteristics and first integrals of the characteristic systems of operator *D*, matricant and free term of the system itself. These components, in turn, have periodic and non-periodic structural components, which are essential in revealing the multiperiodic nature of the movements described by the system under study. The representation of a solution with the selected multiperiodic components is called the multiperiodic structure of the solution. It is realized on the basis of the well-known Bohr's theorem on the connection of a periodic function of many variables and a quasiperiodic function of one variable. Thus, more specifically, the multiperiodic structures of general and multiperiodic solutions of homogeneous and inhomogeneous systems with perturbed input data are investigated. In this spirit, the zeros of the operator D and the matricant of the system are studied. The conditions for the absence and existence of multiperiodic solutions of both homogeneous and inhomogeneous systems are established.

Keywords: multiperiodic solutions, autonomous system, operator of differentiation, Lyapunov's vector field, perturbation.

1. Introduction. The foundations of the method used in this note were laid in [1, 2], which were further developed in [3–10] and applied to the study of solutions different problems in the partial differential equations [11, 12]. These methods with simple modifications extend to the study solutions of problems of the differential and integro-differential equations of different types [1-12], in particular, problems on multi-frequency solutions of equations from control theory [13]. The methods of research for multiperiodic solutions are successfully combined by methods for studying solutions of boundary value problems for equations of mathematical physics. Elements of the methods of [1, 2] can easily be found in [14,15], where time-oscillating solutions of boundary value problems are studied by the parameterization method.

As noted above, the considered system of partial differential equations along with multidimensional time contains space independent variables, according to which differentiation is carried out to the directions of the different vector fields. The autonomous case of this system was considered in [11, 12], where differentiation with respect to time variables was carried out in the direction of the main diagonal of space, and the free term of the system was independent of time variables. In this case, these parameters of the systems received perturbations depending on time variables. In the note, the method for studying multiperiodic structures of general and multiperiodic solutions is developed, the conditions for the existence of a multiperiodic solution are established, and its integral representation is given.

We consider the system of linear equations

$$Dx = Ax + f(\tau,t,\zeta) \tag{1.1}$$

with differentiation operator

$$D = \frac{\partial}{\partial\tau} + \left\langle a, \frac{\partial}{\partial t}\right\rangle + \left\langle \nu I\zeta + g, \frac{\partial}{\partial\zeta}\right\rangle, \tag{1.2}$$

where $\tau \in R$, $t = (t_1, ..., t_m) \in R^m$, $\zeta = (\zeta_1, ..., \zeta_l) \in R_\delta^{2l}$, $\zeta_j = (\xi_j, \eta_j) \in R_\delta^2$, $j = \overline{1, j}$, $R_\delta^2 = \{\zeta_j \in R_\delta^2 : |\zeta_j| = \sqrt{\xi_j^2 + \eta_j^2} < \delta, j = \overline{1,l}\}$, $\delta = const > 0$ are independent variables with areas of change; $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m}\right)$ и $\frac{\partial}{\partial\zeta} = \left(\frac{\partial}{\partial\zeta_1}, ..., \frac{\partial}{\partial\zeta_l}\right)$, $\frac{\partial}{\partial\zeta_j} = \left(\frac{\partial}{\partial\xi_j}, \frac{\partial}{\partial\eta_j}\right)$, $j = \overline{1,l}$ are vector differentiation operators; $I = diag\,(I_2, ..., I_2)$ is a matrix with l-blocks, I_2 is symplectic unit of the second order, $\nu = (\nu_1, ..., \nu_l)$ is a constant vector, $\nu I = diag\,(\nu_1 I_2, ..., \nu_l I_2)$, $a = (a_1(\tau,t), ..., a_m(\tau,t)) = a(\tau,t)$, $g = (g_1(\tau), ..., g_l(\tau)) = g(\tau)$ are vector functions, $\langle\ ,\ \rangle$ is the sign of the scalar product of vectors; A is a constant $n \times n$-matrix, $f = f(\tau,t,\zeta)$ is n-vector-function of variables $(\tau, t, \zeta) \in R \times R^m \times R_\delta^{2l}$.

The main objective of this note is to determine the multiperiodic structures of solutions of the problems (1.1) - (1.2).

2. Multiperiodic structure of zeros of the differentiation operator D. We introduce the equation

$$Du = 0 \tag{2.1}$$

with the required scalar function $u = u(\tau,t,\zeta)$ and the initial condition

$$u\big|_{\tau=\tau^0} = v(t) \in C_t^{(e)}(R^m), \tag{2.1'}$$

where D is the differentiation operator with respect to (τ,t,ζ) of the form (1.2). The solutions of equation (2.1) are called the zeros of the operator D.

Suppose that 1) the vector function $a(\tau,t)$ has the property of smoothness with respect to $(\tau,t) \in R \times R^m$ of order $(0,e) = (0,1,...,1)$:

$$a(\tau+\theta, t+q\omega) = a(\tau,t) \in C_{\tau,t}^{(0,e)}(R \times R^m),\ q \in Z^m, \tag{2.2}$$

2) positive constants $\nu_1, ..., \nu_l$ are rationally incommensurable:

$$q_i\nu_i + q_j\nu_j \neq 0,\ \ q_i^2 + q_j^2 \neq 0,\quad q_i, q_j \in Z,\ (i,\ j = \overline{0,l}), \tag{2.3}$$

therefore, numbers $\alpha_j = 2\pi\nu_j^{-1}$, $j = \overline{1,l}$ are also incommensurable,

3) vector-functions $g_j(\tau) = (\varphi_j(\tau), \psi_j(\tau))$, $j = \overline{1,l}$ are continuous and β_j- periodic:

$$g_j(\tau+\beta_j) = g_j(\tau) \in C_\tau^{(0)}(R),\ j = \overline{1,l}, \tag{2.4}$$

where α_k, $k = \overline{1,l}$ and β_j, $j = \overline{1,l}$ are incommensurable positive constants.

It follows from condition (2.2) that the vector field $\dot{t} = a(\tau,t)$ determines the characteristic $t = \lambda(\tau,\tau^0,t^0)$, emanating from any initial point $(\tau^0, t^0) \in R \times R^m$, and moreover, it has the properties that are known from [2].

Lemma 2.1. *Let condition (2.2) be satisfied. Then under the condition* $v(t+q\omega)=v(t)\in C_t^{(e)}(R^m),\ q\in Z^m$ *the zeros* $u(\tau^0,\tau,t)=v(\lambda(\tau^0,\tau,t))$ *of the operator* D *with the initial data (2.1') have the multiperiodicity property of the form*

$$u(\tau^0+\theta,\tau+\theta,t+q\omega)=u(\tau^0,\tau,t),\ q\in Z^m.$$

The vector fields

$$\dot{\zeta}=\nu I\zeta+g(\tau) \tag{2.5}$$

determines the characteristic

$$\zeta=Z(\tau-\tau^0)[\zeta^0-z(\tau^0)]+z(\tau), \tag{2.6}$$

where $Z(\tau)=diag\ [Z_1(\tau),...,Z_l(\tau)],\ z(\tau)=(z_1(\tau),...,z_l(\tau)),\ \zeta^0=(\zeta_1^0,...,\zeta_l^0)$. Then we have the first integral of equation (2.5)

$$\zeta^0=Z(\tau^0-\tau)[\zeta-z(\tau)]+z(\tau^0)\equiv\mu(\tau^0,\tau,\zeta) \tag{2.7}$$

By virtue to the connection between $\sigma_j=\sigma_j(\tau)$ and $h_j=h_j(s_j,\sigma_j)$ of the form $\sigma_j(\tau)=h_j(\tau,\tau),\ \dfrac{d\sigma_j}{d\tau}=\dfrac{dh_j(\tau,\tau)}{d\tau}=\dfrac{\partial h_j(s_j,\sigma_j)}{\partial s_j}+\dfrac{\partial h_j(s_j,\sigma_j)}{\partial\sigma_j}$ with $\sigma_j=s_j=\tau$ leads to a transition from the differentiation operator D to the differentiation operator

$$\overline{D}=\frac{\partial}{\partial\tau}+\left\langle a(\tau,t),\frac{\partial}{\partial t}\right\rangle+\left\langle e,\frac{\partial}{\partial s}\right\rangle+\left\langle e,\frac{\partial}{\partial\sigma}\right\rangle+\left\langle\nu I h+g(\sigma),\frac{\partial}{\partial h}\right\rangle+\left\langle\frac{\partial h}{\partial s}+\frac{\partial h}{\partial\sigma},\frac{\partial}{\partial h}\right\rangle, \tag{2.8}$$

where $s=(s_1,...,s_l),\ \sigma=(\sigma_1,...,\sigma_l),\ g(\sigma)=(g_1(\sigma_1),...,g_l(\sigma_l)),\ e=(1,...,1)$ – l-vector, $h=(h_1,...,h_l),\ h_j=h_j(s_j,\sigma_j),\ j=\overline{1,l},\ \dfrac{\partial h}{\partial s}=\left(\dfrac{\partial h_1}{\partial s_1},...,\dfrac{\partial h_l}{\partial s_l}\right),\ \dfrac{\partial h}{\partial\sigma}=\left(\dfrac{\partial h_1}{\partial\sigma_1},...,\dfrac{\partial h_l}{\partial\sigma_l}\right).$

Lemma 2.2. *Let conditions (2.3) and (2.4) be satisfied. Then the zeros* $u(\tau^0,\tau,\zeta)=w(\mu(\tau^0,\tau,\zeta))$ *of the operator* D *with the initial condition* $u|_{\tau=\tau^0}=w(\zeta)\in C_\zeta^{(e)}(R^l)$ *have a multiperiodic structure of the form* $\overline{u}(s^0,s,\sigma,\zeta)=w(h(s^0-s,z(s^0),\zeta-z(\sigma)))$ *with the vector function* $h(s-s^0,z(\sigma),\ \zeta^0-z^0)=Z(s-s^0)[\zeta^0-z(s^0)]+z(\sigma)$*, at that*

$$\begin{gathered}\overline{u}(s^0,s,\sigma,\zeta)\Big|_{\sigma=s=\tau\tilde{e},\ s^0=\tau^0\tilde{e}}=u(\tau^0,\tau,\zeta),\\ h(\tilde{e}\tau^0-\tilde{e}\tau,z(\tilde{e}\tau^0),\zeta-z(\tilde{e}\tau))=\mu(\tau^0,\tau,\zeta).\end{gathered} \tag{2.9}$$

The following theorem is proved on the bases of these Lemmas 2.1 and 2.2.

Theorem 2.1. *Let conditions (2.2) - (2.4) be satisfied. Then the solution* $u(\tau^0,\tau,t,\zeta)$ *of equation (2.1) with the initial condition* $u|_{\tau=\tau^0}=u^0(t,\zeta)\in C_{t,\zeta}^{(e,\tilde{e})}(R^m\times R^l)$ *is determined by the relation* $u(\tau^0,\tau,t,\zeta)=u^0(\lambda(\tau^0,\tau,t),\mu(\tau^0,\tau,\zeta))$, *which under the conditions* $\lambda(\tau^0,\tau+\theta,t)=\lambda(\tau^0,\tau,t)$ *and* $u^0(t+q\omega,\zeta)=u^0(t,\zeta),\ q\in Z^m$ *has a multiperiodic structure with respect to* (τ,t,s,σ) *with period* $(\theta,\omega,\alpha,\beta)$ *of the form*

$$\overline{u}(\tau^0,\tau,t;s^0,s,\sigma,\zeta)=u^0(\lambda(\tau^0,\tau,t),h(s^0-s,z(s^0),\zeta-z(\sigma))),$$

where the vector-function $h(s,z,\zeta)$ *has the form* $h(s-s^0, z(\sigma), \zeta^0 - z^0) = Z(s-s^0)[\zeta^0 - z(s^0)] + z(\sigma)$, $\hat{e} = (1,\dots,1)$ *is* m*-vector,* $\tilde{e} = (1,\dots,1)$ *is* l*-vector, moreover* $\bar{u}\big|_{\sigma=s=\tilde{e}\tau,\ s^0=\tilde{e}\tau^0} = u(\tau^0,\tau,t,\zeta)$.

3. The multiperiodic structure of the solution of a homogeneous linear D-system with constant coefficients. We consider a homogeneous linear system

$$Dx = Ax \tag{3.1}$$

with a differentiation operator D of the form (1.2) and a constant $n \times n$-matrix A.

We will put the problem of determining the multiperiodic structure of the solution x of the system (3.1) with the initial condition

$$x\big|_{\tau=\tau^0} = u(t,\zeta) \in C_{t,\zeta}^{(\hat{e},\tilde{e})}\left(R^m \times R^l\right). \tag{3.1°}$$

To this end, we begin the solution of the problem by studying the multiperiodic structure of the matricant

$$X(\tau) = \exp[A\tau] \tag{3.2}$$

of the system (3.1). We need the following lemmas, to do this, which are given without proof.

Lemma 3.1. *If* $f_j(\tau+\theta_j) = f_j(\tau),\ j=\overline{1,r}$ *is some collection of the periodic functions with rationally commensurate periods:* $\theta_j\theta_k^{-1} = r_{jk}$ *is a rational number for* $j,k=\overline{1,r}$, *then for these functions exist a common period* θ*:* $f_j(\tau+\theta) = f_j(\tau),\ j=\overline{1,r}$.

Lemma 3.2. *If the real parts of all eigenvalues equal to zero and all the elementary divisors are simple of the constant matricant* $Y(\tau) = \exp[I\tau]$, *then all the elements of the matrix* I *are periodic functions*.

We consider the multiperiodic matrix $T(\hat{\tau}) = T(\tau_1,\dots,\tau_\rho)$ with period $\gamma = (\gamma_1,\dots,\gamma_\rho)$, where $\gamma_1,\dots,\gamma_\rho$ are rationally incommensurable constants. Since $(\partial/\partial\tau_k)Y_{j_k}(\tau_k) = J_j Y_{j_k}(\tau_k)$, the matrix $T(\hat{\tau})$ satisfies the equation

$$\hat{D}T(\hat{\tau}) = I\,T(\hat{\tau}), \tag{3.3}$$

where the operator $\hat{D}$ is determined by

$$\hat{D} = \left\langle \hat{e}, \frac{\partial}{\partial\hat{\tau}} \right\rangle = \frac{\partial}{\partial\tau_1} + \dots + \frac{\partial}{\partial\tau_\rho}, \tag{3.4}$$

$\hat{e} = (1,\dots,1)$ is a ρ-vector. Obviously, under $\hat{\tau} = \hat{e}\tau$ we have $T(\hat{e}\tau) = Y(\tau)$ and

$$\dot{Y}(\tau) = \dot{T}(\hat{e}\tau) = I\,T(\hat{e}\tau) = I\,Y(\tau). \tag{3.5}$$

Thus, the multiperiodic matrix $T(\hat{\tau})$ defines the multiperiodic structure of the matricant $Y(\tau)$

$$Y(\tau) = T(\tau_1,\dots,\tau_\rho)\big|_{\tau_1=\dots=\tau_\rho=\tau}. \tag{3.6}$$

Lemma 3.3. *The matricant* $Y(\tau)$ *of the system (3.5) under the conditions of Lemma 3.2 has a multi-periodic structure in the form of a matrix* $T(\hat{\tau}) = T(\tau_1,\dots,\tau_\rho)$ *which satisfies the system (3.3) with the differentiation operator (3.4) and along the characteristics* $\hat{\tau} = \hat{e}\tau$ *of the operator* $\hat{D}$ *turns into* $Y(\tau)$, *in other words, these matrices are related by the relation (3.6)*

Indeed, we making the replacement $X = Y(\tau)Z$ in the equation

$$\dot{X} = AX \tag{3.7}$$

obtain the equation $\dot{Z} = Y^{-1}(\tau)[AY(\tau) - \dot{Y}(\tau)]Z$. Therefore, according to Lemma 3.3, the multiperiodic structure of the matricant (3.2), by virtue of equality $X(\tau) = Y(\tau) \cdot Z(\tau)$, is determined by a matrix $\widehat{X}(\tau, \widehat{\tau})$ of the form

$$\widehat{X}(\tau, \widehat{\tau}) = X(\tau, \tau_1, ..., \tau_\rho) = T(\tau_1, ..., \tau_\rho) e^{R\tau}, \tag{3.8}$$

which is connected by the matricant $X(\tau)$, by relation

$$\widehat{X}(\tau, \widehat{\tau})\Big|_{\widehat{\tau} = \widehat{e}\,\tau} = X(\tau). \tag{3.9}$$

Theorem 3.1. *In the presence of complex eigenvalues of the matrix* A*, the matricant (3.2) of the system (3.7) has a multiperiodic structure defined by the matrix (3.8) and relations (3.3) - (3.6), and it along the characteristics* $\widehat{\tau} = \widehat{e}\,\tau$ *of the operator* $\widehat{D}$ *satisfies condition (3.9). The matrix* $T(\widehat{\tau})$ *turns into a constant matrix in the absence of complex eigenvalues.*

Now the solution of the objectives set can be formulated as Theorem 3.2.

Theorem 3.2. *Let conditions (2.2) - (2.4) be satisfied. Then the solution* $x(\tau^0, \tau, t, \zeta)$ *of the problem (3.1) - (3.1°) defined by relation*

$$x(\tau^0, \tau, t, \zeta) = X(\tau) u(\lambda(\tau^0, \tau, t), \mu(\tau^0, \tau, \zeta)) \tag{3.10}$$

has a multi-periodic structure in the form of a vector-function

$$\widehat{x}(\tau^0, \tau, \widehat{\tau}, t, s^0 s, \sigma, \zeta) = \widehat{X}(\tau, \widehat{\tau}) \overline{u}(\lambda(\tau^0, \tau, t), h(s^0 - s, z(s^0), \zeta - z(\sigma))), \tag{3.11}$$

that satisfies equation

$$\overline{\overline{D}} = A\widehat{x} \tag{3.12}$$

with the differentiation operator

$$\overline{\overline{D}} = \overline{D} + \widehat{D}, \tag{3.13}$$

defined by relations (2.8) and (3.4).

Proof. The representation (3.10) is known from [2], and (3.11) follows from the proved Theorems 2.1 and 3.1. The identity (3.12) can be verified by a simple check.

Theorem 3.3. *Under the conditions of the Theorem 3.2, the system (3.1) allowed nonzero multiperiodic solutions enough for the matrix* A *to have at least one eigenvalue* $\lambda = \lambda(A)$ *with the real part* $\operatorname{Re} \lambda(A) = 0$ *equal to zero.*

The theorem could be proved on the basis of a similar theorem from the theory of the systems of ordinary differential equations.

We have the following theorem from the theorem 3.3, as a corollary.

Theorem 3.4. *Under the conditions of the Theorem 3.3, the system (3.1) did not admit the multiperiodic solution other than trivial, it is sufficient that all eigenvalues of the matrix* A *have nonzero real parts.*

The general solution x of the system (3.1) can be represented in the form

$$x(\tau, t, \zeta) = X(\tau) u(\tau, t, \zeta), \tag{3.14}$$

where $u = u(\tau, t, \zeta)$ is the zero of the operator D with the general initial condition for $\tau = 0$: $x(0, t, \zeta) = u(0, t, \zeta) = u_0(t, \zeta)$, $X(\tau) = \exp[A\tau]$ is the matricant of the system.

Theorem 3.5. . *Under the conditions (2.2) - (2.4), the system (3.1) had* (θ, ω) *-periodic with respect to* (τ, t) *solutions of the form (3.14) corresponding to the multiperiodic zero of the operator* D *with the same periods, it is necessary and sufficient that the monodromy matrix* $X(\theta)$ *satisfies condition*

$$\det[X(\theta) - E] = 0. \tag{3.15}$$

Proof. Under the conditions of the theorem, its justice is equivalent to the solvability of equation $X(\tau+\theta)u = X(\tau)u$ in the space of (θ,ω)-periodic with respect to (τ,t) zeros $u = u(\tau,t,\zeta)$ of the operator D.

We arrive at the solvability of the system of equations $[X(\theta)-E]u = 0$, which is equivalent to the condition (3.15) taking into account the properties of the matricant $X(\tau+\theta) = X(\tau)X(\theta)$ from the system $X(\tau+\theta)u = X(\tau)u$.

In conclusion, we note that the fulfillment of condition

$$\det[X(\theta)-E] \neq 0 \tag{3.16}$$

guarantees the absence of such solutions.

Theorem 3.6. *Let conditions (2.2) - (2.4) and (3.16) be satisfied. Then the system (3.1) allowed nonzero (θ,ω)-periodic solutions of the form (3.14) necessary and sufficient for the functional-difference equations*

$$u(\tau+\theta, t+q\omega,\zeta) = [X(\theta)-E]^{-1}X(\theta)[u(\tau+\theta,t+q\omega,\zeta)-u(\tau,t,\zeta)],\ q\in Z^m \tag{3.17}$$

to be solvable in the space of zeros of the operator D.

Proof. Under the condition (3.16) from the definition of (θ,ω)-periodicity with respect to (τ,t) of solution (2.7), we have the equation (3.17). We must be to take into account that $u(\tau,t,\zeta)$ is the zero of the operator D to complete the proof. If the equation (3.17) has only zero solutions, then, under the condition (3.16), the system (3.1) does not have a nontrivial multiperiodic solution.

4. The multiperiodic structure of an inhomogeneous linear system with operator *D*. Consider the inhomogeneous linear equation (1.1) corresponding to the homogeneous equation (3.1), where the n-vector function $f(\tau,t,\zeta)$ satisfies condition

$$f(\tau+\theta,t+q\omega,\zeta) = f(\tau,t,\zeta) \in C^{(0,\hat{e},\tilde{e})}_{\tau,t,\zeta}(R\times R^m\times R^l). \tag{4.1}$$

Assume that the condition (3.16) is fulfilled and we search for the (θ,ω)-periodic with respect to (τ,t) solution $x(\tau,t,\zeta)$ of the system (1.1) that corresponds to zero $u(\tau,t,\zeta)$ of the operator D possessing the property of multiperiodicity with the same periods (θ,ω) for (τ,t).

Therefore, we have the solution

$$x(\tau,t,\zeta) = X(\tau)u(\tau,t,\zeta) + X(\tau)\int_0^\tau X^{-1}(s)f(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))ds \tag{4.2}$$

with zero $u(\tau+\theta,t+q\omega,\zeta) = u(\tau,t,\zeta),\ q\in Z^m$ of the operator D having the property $x(\tau+\theta,t+q\omega,\zeta) = x(\tau,t,\zeta),\ q\in Z^m$.

By accepting the notation based on (4.2)

$$f_\theta(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta)) = \begin{cases} f(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta)),\ \tau \xrightarrow{s} 0, \\ f(s,\lambda(s,\tau+\theta,t),\mu(s,\tau+\theta,\zeta)),\ 0 \xrightarrow{s} \tau+\theta, \end{cases}$$

where $\gamma \xrightarrow{s} \delta$ means changes in the variable s from γ to δ, the multiperiodic solutions can be presented in compact form

$$x(\tau,t,\zeta) = [X^{-1}(\tau+\theta) - X^{-1}(\tau)]^{-1}\int_\tau^{\tau+\theta} X^{-1}(s)f_\theta(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))ds. \tag{4.3}$$

Obviously, if the system (3.1) does not have multiperiodic solutions, except for zero, then the solution (4.3) of the system (1.1) is a unique multiperiodic solution.

Further, we have solutions

$$\hat{x}(s,\sigma,\hat{\tau},\tau,t,\zeta)=\left[\hat{X}^{-1}(\tau+\theta,\hat{\tau}+\hat{e}\theta)-\hat{X}^{-1}(\tau,\hat{\tau})\right]^{-1}\int_{\tau}^{\tau+\theta}X^{-1}(\varepsilon)f_{\theta}(\varepsilon,\lambda(\varepsilon,\tau,t),h(\varepsilon-s,z(\varepsilon),\zeta-z(\sigma)))d\varepsilon \quad (4.4)$$

of the equation

$$\overline{\overline{D}}\hat{x}=A\hat{x}+f(\tau,t,\zeta) \quad (4.5)$$

with the differentiation operator (3.13) from representation (4.3) on the basis of multiperiodic structures (2.9) and (3.8) of the quantity $\mu(s,\tau,\zeta)$ and $X(\tau)$.

Теорема 4.1. *Assume that conditions (2.2) - (2.4), (3.16) and (4.1) are satisfied, and the homogeneous system (3.1) does not have multiperiodic solutions except zero. Then the system (1.1) has a unique* (θ,ω)*-periodic solution (4.3) for which the* $(\alpha,\beta,\gamma,\theta,\omega)$*-periodic with respect to* $(s,\sigma,\hat{\tau},\tau,t)$ *structure (4.4) satisfies equation (4.5) with the differentiation operator (3.13).*

In conclusion, note that we can derive the multiperiodic structure of the general solution (4.2) of the system (1.1) similarly to formula (4.4).

Conclusion. A method for studying the multiperiodic structure of oscillatory solutions of perturbed linear autonomous systems of the form (1.1) - (1.2) was developed. The main essence of the method for studying the multiperiodic structures of solution of the system under consideration is a combination of the known methods [1-3] with the methods used in [11, 12] for the autonomous systems. In conclusion, the sufficient conditions for the existence of the multiperiodic solutions of linear systems (1.1) - (1.2) with the differentiation operator *D* in the directions of a toroidal vector field with respect to time variables and of the form of Lyapunov's systems with respect to space variables were established. Moreover, relation (4.3) is an integral representation of the multiperiodic solution of the system, and (4.4) determines its multiperiodic structure.

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ВЕКТОРЛЫҚ ӨРІС БОЙЫНША ДИФФЕРЕНЦИАЛДАУ ОПЕРАТОРЛЫ ҚОЗДЫРЫЛҒАН СЫЗЫҚТЫ АВТОНОМДЫҚ ЖҮЙЕЛЕРДІҢ КӨППЕРИОДТЫ ШЕШІМДЕРІН ЗЕРТТЕУ

Аннотация. Тәуелсіз кеңістік айнымалысына қатысты Ляпунов жүйесі түріндегі және уақыт айнымалысына қатысты, көппериодты тороидалды түрдегі векторлық өрістер бағыты бойынша *D* дифференциалдау операторлы сызықты жүйе қарастырылады. Жүйені анықтайтын барлық берілген өлшемдер уақыт айнымалысынан көппериодты тәуелді, не олардан тәуелсіз болады. Бұл жағдайда жүйені анықтайтын кейбір берілгендерге уақыт айнымалысынан тәуелді қоздыртқы берілген. Рационалды өлшенбейтін жиіліктердің жекеленген периодты қозғалыстарының суперпозициясы түріндегі жүйе арқылы сипатталған ізделінді қозғалыс туралы сұрақ зерттеледі. Бастапқы есептер және қозғалыстардың көппериодтылығы туралы есептер зерттеледі. Есептің шешімін анықтау кезінде жүйенің бастапқы нүктеден шығатын характеристика маңайында интегралданатыны, одан кейін бастапқы берілгендер характеристикалық жүйенің бірінші интегралдарымен ауыстырылатыны белгілі. Сонымен, ізделінді шешім келесі компоненттерден тұрады: *D* операторының характеристикалық жүйесінің характеристикасы мен бірінші интегралдары, жүйенің бос мүшесі мен матриcанты. Бұл компоненттердің зерттелуші жүйемен сипатталған қозғалыстың көппериодтылық табиғатын ашу кезінде маңызды мағынасы бар болатын периодты және периодты емес құрылымдық құраушылары болады. Шешімді ерекшеленген көппериодты құраушылар арқылы сипаттауды шешімнің көппериодтылық құрылымы деп атайды. Ол көп айнымалы периодты функциялар мен бір айнымалы квазипериодты функцияларының байланысы туралы Бордың танымал теоремасы негізінде жүзеге асады. Сонымен, жүйелерді анықтайтын берілгендері қоздырылған жағдайды біртекті және біртексіз жүйелердің жалпы және көппериодты шешімдерінің көппериодты құрылымы нақты зерттелген. Осылайша *D* операторының нөлдері мен жүйенің матриcанты зерттелген. Біртекті және біртексіз жүйелердің көппериодты шешімдерінің бар болу және болмау шарттары тағайындалған.

Түйін сөздер: көппериодты шешім, автономдық жүйе, дифференциалдау операторы, Ляпунов векторлық өрісі, қоздыртқы.

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ИССЛЕДОВАНИЕ МНОГОПЕРИОДИЧЕСКИХ РЕШЕНИЙ ВОЗМУЩЕННЫХ ЛИНЕЙНЫХ АВТОНОМНЫХ СИСТЕМ С ОПЕРАТОРОМ ДИФФЕРЕНЦИРОВАНИЯ ПО ВЕКТОРНОМУ ПОЛЮ

Аннотация. Рассматривается линейная система с оператором дифференцирования D по направлениям векторных полей вида системы Ляпунова относительно пространственных независимых переменных и многопериодического тороидального вида относительно временных переменных. Все входные данные системы либо многопериодично зависят от временных переменных, либо от них не зависят. В данном случае некоторые входные данные получили возмущения, зависящие от временных переменных. Исследуется вопрос о представлении искомого движения, описанного системой в виде суперпозиции отдельных периодических движений рационально несоизмеримых частот. Изучаются начальные задачи и задачи о многопериодичности движений. Известно, что при определении решений задач система интегрируется вдоль характеристик, исходящих из начальных точек, а затем начальные данные заменяются первыми интегралами характеристических систем. Таким образом, искомое решение состоит из следующих компонентов: характеристик и первых интегралов характеристических систем оператора D, матрицанта и свободного члена самой системы. Эти компоненты, в свою очередь, имеют периодические и непериодические структурные составляющие, которые имеют существенное значение при раскрытии многопериодической природы движений, описанных исследуемой системой. Представление решения с выделенными многопериодическими составляющими названо многопериодической структурой решения. Оно реализуется на основе известной теоремы Бора о связи периодической функции от многих переменных и квазипериодической функции одной переменной. Таким образом, более конкретно исследуются многопериодические структуры общих и многопериодических решений однородных и неоднородных систем с возмущенными входными данными. В таком духе изучаются нули оператора D и матрицант системы. Устанавливаются условия отсутствия и существования многопериодических решений как однородных, так и неоднородных систем.

Ключевые слова: многопериодическое решение, автономная система, оператор дифференцирования, Ляпунова векторное поле, возмущение.

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REFERENCES

[1] Kharasakhal V.Kh. **(1970)** Almost-periodic solutions of ordinary differential equations [Pochti-periodicheskie resheniia obyknovennykh differentsialnykh uravnenii]. Alma-Ata, Nauka (in Russian).

[2] Umbetzhanov D.U. **(1979)** Almostmultiperiodic solutions of partial differential equations [Pochti mnogoperiodicheskie resheniia differentsialnykh uravnenii v chastnykh proizvodnykh]. Alma-Ata: Nauka (In Russian).

[3] Kulzhumieva A.A., Sartabanov Zh.A. **(2013)** Periodic solutions of the systems of differential equations with multidimensional time [Periodicheskie resheniia system differentsialnykh uravne-nii s mnogomernym vremenem]. Uralsk: RITs ZKGU (in Russian). ISBN 978-601-266-128-6.

[4] Sartabanov Zh. (1989) Pseudoperiodic solutions of a system of integro differential equations, Ukrainian Mathemstical Journal, 41:1, 116-120. DOI: 10.1007/BF01060661

[5] Kulzhumiyeva A.A., Sartabanov Zh.A. **(2016)** On reducibility of linear D-e-system with constant coefficients on the diagonal to D-e-system with Jordan matrix in the case of equivalence of its higher order one equation//Bulletin of the Karaganda University-Mathematics, 84:4, 88-93.

[6] Kulzhumiyeva A.A., Sartabanov Z.A. **(2017)** On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables, Eurasian Mathematical Journal, 8:1, 67-75.

[7] Kulzhumiyeva A.A., Sartabanov Zh.A. **(2017)** Reduction of linear homogeneous D-e-systems to the Jordan canonical form, News of the National Academy of Sciences of the Republic of Kazakhstan. Series Physico-Mathematical, 5:315, 5-12.

[8] Sartabanov Z.A. **(2017)** The multi-period solution of a linear system of equations with the operator of differentiation along the main diagonal of the space of independent variables and delayed arguments, AIP Conference Proceedings, 1880, 040020. DOI: 10.1063/1.5000636

[9] Kulzhumiyeva A.A., Sartabanov Zh.A. **(2018)** General bounded multiperiodic solutions of linear equations with differential operator in the direction of the mail diagonal, Bulletin of the Karaganda University-Mathematics, 92:4, 44-53. DOI:10.31489/2018M4/44-53

[10] Kulzhumiyeva A.A., Sartabanov Z.A. **(2019)** Integration of a linear equation with differential operator, corresponding to the main diagonal in the space of independent variables, and coefficients, constant on the diagonal, Russian Mathematics, 63: 6, 29-41. DOI:10.3103/S1066369X19060045

[11] Sartabanov Z.A., Omarova B.Z. **(2018)** Multiperiodic solutions of autonomous systems with operator of differentiation on the Lyapunov’s vector field, AIP Conference Proceedings, 1997, 020041-1 - 020041-4. DOI:10.1063/1.5049035

[12] Sartabanov Zh.A., Omarova B.Zh.(2019) On multi-periodic solutions of quasilinear autonomous systems with operator of differentiation on the Lyapunov's vector field, Bulletin of the Karaganda University-Mathematics, 94:2, 70-83.DOI:10.31489/2019M2/70-83

[13] Kerimbekov A., Abdyldaeva E. **(2016)** On the solvability of a nonlinear tracking problem under boundary control for the elastic oscillations described by fredholmintegro-differential equations, IFIP Advances in Information and Communication Technology, 494, 312-321. DOI: 10.1007/978-3-319-55795-3_29

[14] Assanova A.T., Alikhanova B.Zh., Nazarova K.Zh. Well-posedness of a nonlocal problem with integral conditions for third order system of the partial differential equations. News of the National Academy of Sciences of the Republic of Kazakhstan.Physico-Mathematical Series.Volume 5, Number 321 (2018), PP. 33-41. DOI: 10.32014/2018.2518-1726.5 ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

[15] Dzhumabaev D.S., Bakirova E.A., Kadirbayeva Zh.M. An algorithm for solving a control problem for a differential equation with a parameter.News of the National Academy of Sciences of the Republic of Kazakhstan.Physico-Mathematical Series. Volume 5, Number 321 (2018), PP. 25-32. DOI: 10.32014/2018.2518-1726.4 ISSN 2518-1726 (Online), ISSN 1991-346X (Print)