NEWS
OF THENATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X
Volume 2, Number 336 (2021), 24 – 32 **https://doi.org/10.32014/2021.2518-1726.17**

UDC 514.753

D. Kurmanbayev[1, 2], K. Yesmakhanova[3]

[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan;
[2]Suleyman Demirel University, Kaskelen, Kazakhstan;
[3]L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan.
E-mail: kurmanbaev.damir@gmail.com

SOLITON DEFORMATION OF INVERTED CATENOID

Abstract. The minimal surface (see [1]) is determined using the Weierstrass representation in three-dimensional space. The solution of the Dirac equation [2] in terms of spinors coincides with the representations of this surface with conservation of isothermal coordinates. The equation represented through the Dirac operator, which is included in the Manakov's L, A, B triple [3] as equivalent to the modified Veselov-Novikov equation (mVN) [4]. The potential U of the Dirac operator is the potential of representing a minimal surface. New solutions of the mVN equation are constructed using the pre-known potentials of the Dirac operator and this algorithm is said to be Moutard transformations [5]. Firstly, the geometric meaning of these transformations which found in [6], [7], gives us the definition of the inversion of the minimal surface, further after finding the exact solutions of the mVN equation, we can represent the inverted surfaces. And these representations of the new potential determine the soliton deformation [8], [9]. In 2014, blowing-up solutions to the mVN equation were obtained using a rigid translation of the initial Enneper surface in [6]. Further results were obtained for the second-order Enneper surface [10]. Now the soliton deformation of an inverted catenoid is found by smooth translation along the second coordinate axis.

In this paper, in order to determine catenoid inversions, it is proposed to find holomorphic objects as Gauss maps and height differential [11]; the soliton deformation of the inverted catenoid is obtained; particular solution of modified Karteweg-de Vries (KdV) equation is found that give some representation of KdV surface [12],[13].

Keywords: Modified Veselov-Novikov equation, Dirac operator, Gauss maps, height differential, stereographic projection, soliton deformation, Moutard transformations, catenoid.

1. Preliminaries. The minimal surface (see [1]) is determined using the Weierstrass representation in three-dimensional space. The introduction to this representation is proposed in the following lemma:

Lemma 1. If $\varphi: D \to \mathbb{C}^3$- is a vector function that satisfies the following conditions:

1. φ - is holomorphic function;
2. $$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0, \tag{1}$$

then there exists a minimal surface $r: D \to \mathbb{R}^3$ for isothermal coordinates

$$\varphi = \frac{\partial r}{\partial z} = (u_z^1, u_z^2, u_z^3).$$

The problem of constructing minimal surfaces is to find functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ that satisfy equation (1). And the general solution of equation (1) is represented through some holomorphic functions $\psi_1, \bar{\psi}_2$ in the following form:

$$\varphi_1 = \frac{i}{2}(\psi_1^2 + \bar{\psi}_2^2), \varphi_2 = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), \varphi_3 = \psi_1\bar{\psi}_2. \tag{2}$$

Now it could be found all the components u^1, u^2, u^3 of minimal surfaces by the Weierstrass representations [2]. For example, catenoid $\psi: \mathcal{U} \to \mathbb{R}^3$ constructed by the following Weierstrass representations:

$$u^1(x,y) = -chx\sin y,$$
$$u^2(x,y) = chx\cos y, \quad (3)$$
$$u^3(x,y) = x.$$

Gauss map is written in terms of $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, as well as the solution of the following Dirac equation [2]:

$$\mathcal{D}\psi = 0, (4)$$

where ψ- are called *spinors.* And

$$\mathcal{D} = \begin{pmatrix} U & \frac{\partial}{\partial z} \\ -\frac{\partial}{\partial \bar{z}} & U \end{pmatrix}$$

-Dirac operator with real-valued potential U.

Likewise the solution of the Dirac equation in terms of spinors

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

coincides with the representations of minimal surface with conservation of isothermal coordinates. Because of this notation catenoid could be given by $\psi_1 = \frac{1}{\sqrt{2}}e^{-\frac{z}{2}}, \psi_2 = \frac{1}{\sqrt{2}}e^{\frac{\bar{z}}{2}}$.

The equation represented through the Dirac operator is included in the Manakov's L, A, B triple [3], which is equivalent (will be discussed below) to the modified Veselov-Novikov equation (mVN) [4]. The potential of the Dirac operator is the potential of representing a minimal surface. New solutions of the mVN equation are constructed using the pre-known potentials of the Dirac operator and this algorithm is said to be Moutard transformations [5]. These transformations could be illustrated in the following form:

$$\mathcal{D}\psi = 0 \to \widetilde{\mathcal{D}}\tilde{\psi} = 0$$

where $\widetilde{\mathcal{D}} = \begin{pmatrix} \widetilde{U} & \frac{\partial}{\partial z} \\ -\frac{\partial}{\partial \bar{z}} & \widetilde{U} \end{pmatrix}$, $\widetilde{U}$ - real-valued function ($z = x + iy$).

To phrase problem statement firstly, we consider the geometric meaning of these transformations (which found in [6],[7]) and definition of the inversion of the minimal surface; further the inverted surfaces could be represented after finding the exact solutions of the mVN equation. And these representations of the new potential determine the soliton deformation of inverted surfaces [8],[9].

Our problem is to analyze the soliton deformation of inverted catenoid by the following items:

1. *Gauss maps, height differential;*
2. *Weierstrass representations;*

Differential of the third coordinate is -

$$du^3 = Re(dh), \quad (5)$$

where dh – is called *height differential* [11].

To understand the geometry of minimal surfaces, we consider the complex-analytic properties of the Gauss map G and dh.

The Gauss map [2] is determined by the formula $G(z) = \frac{\partial r}{\partial z} = \frac{1}{2}(r_u - ir_v)$ and by (2), (5) we obtain

$$G(z) = \left(\frac{i}{2}(\psi_1^2 + \bar{\psi}_2^2), \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), \psi_1\bar{\psi}_2\right), \quad (6)$$

$$dh = \frac{\partial}{\partial z}(\psi_1\bar{\psi}_2)dz + \frac{\partial}{\partial \bar{z}}(\psi_1\bar{\psi}_2)d\bar{z}. \quad (7)$$

Surfaces $\widetilde{\psi}$ constructed by $\psi_1(z,\bar{z},t), \psi_2(z,\bar{z},t)$ (will be found below) using Weierstrass representations determine the soliton deformation of the surface ψ [8],[9].

It is known in [3],[4] that *modified Veselov-Novikov equations* (mVN) -

$$U_t = \left(U_{zzz} + 3U_zV + \frac{3}{2}UV_z\right) + \left(U_{\bar{z}\bar{z}\bar{z}} + 3U_{\bar{z}}\bar{V} + \frac{3}{2}U\bar{V}_{\bar{z}}\right), \tag{8}$$

$$V_{\bar{z}} = (U^2)_z, \tag{9}$$

are represented by Manakov's L, $\mathcal{A}, \mathcal{B}$ triple:

$$\mathcal{D}_t + [\mathcal{D}, \mathcal{A}] - \mathcal{B}\mathcal{D} = 0$$

where $\mathcal{D}$ – Dirac operator and $\mathcal{A}, \mathcal{B}$ – are special differential operators represented by the following forms ([5],[6]):

$$\mathcal{A} = \frac{\partial^3}{\partial z^3} + \frac{\partial^3}{\partial \bar{z}^3} + 3\begin{pmatrix} V & 0 \\ U_z & 0 \end{pmatrix}\frac{\partial}{\partial z} + 3\begin{pmatrix} 0 & -U_{\bar{z}} \\ 0 & \bar{V} \end{pmatrix}\frac{\partial}{\partial \bar{z}} + \frac{3}{2}\begin{pmatrix} V_z & 2U\bar{V} \\ -2UV & \bar{V}_{\bar{z}} \end{pmatrix},$$

$$\mathcal{B} = 3\begin{pmatrix} -V & 0 \\ -2U_z & V \end{pmatrix}\frac{\partial}{\partial z} + 3\begin{pmatrix} \bar{V} & 2U_{\bar{z}} \\ 0 & -\bar{V} \end{pmatrix}\frac{\partial}{\partial \bar{z}} + \frac{3}{2}\begin{pmatrix} \bar{V}_{\bar{z}} - V_z & 2U_{\bar{z}\bar{z}} \\ -2U_{zz} & V_z - \bar{V}_{\bar{z}} \end{pmatrix}.$$

Usually Manakov's L, A, B triple was written in [5],[7] by

$$L_t + [L, A] - BL = 0,$$

in terms of operator

$$L = \begin{pmatrix} \frac{\partial}{\partial z} & -U \\ U & \frac{\partial}{\partial \bar{z}} \end{pmatrix},$$

and Dirac operator given above $\mathcal{D} = L \cdot \Gamma$, where $\Gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

So operators $\mathcal{A}, \mathcal{B}$ obtained by the following formulas [3], [7]:

$$\mathcal{A} = -\Gamma A \Gamma, \mathcal{B} = \Gamma A \Gamma + A + B.$$

If U, V depend on variables x, y, then mVN equations (8),(9) can be rewritten in the following form:

$$U_t = U_{xxx} - 3U_xU_{yy} + \frac{3}{2}U_x(V + \bar{V}) + \frac{3}{4}U(V_x + \bar{V}_x) + \frac{3i}{2}U_y(\bar{V} - V) + \frac{3i}{4}U(\bar{V}_y - V_y), \tag{10}$$

$$V_x - (U^2)_x = -i(V_y + (U^2)_y). \tag{11}$$

Let U, V pre-known solutions of mVN equations (8),(9) and $\Psi_0 = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$ satisfy the following system:

$$\begin{cases} \mathcal{D}\Psi_0 = 0, \\ \Psi_{0t} = \mathcal{A}\Psi_0. \end{cases}$$

Last system lead to the system of linear equations is the following Airy type equations (G.Airy) for pre-known $U = V = 0$ solutions:

$$\frac{\partial \psi_1}{\partial t} = \frac{\partial^3 \psi_1}{\partial z^3}, \frac{\partial \psi_2}{\partial t} = \frac{\partial^3 \psi_2}{\partial \bar{z}^3}, \tag{12}$$

with initial data

$$\psi_1(z, \bar{z}, 0) = \frac{e^{-\frac{z}{2}}}{\sqrt{2}}, \psi_2(z, \bar{z}, 0) = \frac{e^{\frac{\bar{z}}{2}}}{\sqrt{2}}. \tag{13}$$

By the successive approximation methods [14], the following solutions of problem (12), (13) are found:

$$\psi_1(z,\bar{z},t)=\frac{e^{-\frac{z}{2}-\frac{t}{8}}}{\sqrt{2}},\psi_2(z,\bar{z},t)=\frac{e^{\frac{\bar{z}}{2}-\frac{t}{8}}}{\sqrt{2}}.$$

which also satisfy the Dirac equations ($U=V=0$):

$$\frac{\partial\psi_1}{\partial\bar{z}}=\frac{\partial\overline{\psi_2}}{\partial\bar{z}}=0.$$

Inversion for minimal surfaces is obtained by new surface $\tilde{\psi}$ constructed by $\psi_1(z,\bar{z},t),\psi_2(z,\bar{z},t)$ with conservation of isothermal coordinates, and in order to analyze the deformation of this surface, will be found the solution of the Dirac equation by the Moutard transformations [5].

These transformations are also called the Darboux transformations for finding solutions of the following modified Veselov-Novikov equation (mVN):

$$\widetilde{U}_t=\left(\widetilde{U}_{zzz}+3\widetilde{U}_z\widetilde{V}+\tfrac{3}{2}\widetilde{U}\widetilde{V}_z\right)+\left(\widetilde{U}_{\bar{z}\bar{z}\bar{z}}+3\widetilde{U}_{\bar{z}}\overline{\widetilde{V}}+\tfrac{3}{2}\widetilde{U}\overline{\widetilde{V}_{\bar{z}}}\right), \tag{14}$$

where

$$\widetilde{V}_{\bar{z}}=\left(\widetilde{U}^2\right)_z.) \tag{15}$$

Note that solutions of mVN equations will be found in variables x,y, therefore

$$\widetilde{U}_t=\widetilde{U}_{xxx}-3\widetilde{U}_x\widetilde{U}_{yy}+\tfrac{3}{2}\widetilde{U}_x\big(\widetilde{V}+\overline{\widetilde{V}}\big)+\tfrac{3}{4}\widetilde{U}\big(\widetilde{V}_x+\overline{\widetilde{V}}_x\big)+\tfrac{3i}{2}\widetilde{U}_y\big(\overline{\widetilde{V}}-\widetilde{V}\big)+\tfrac{3i}{4}\widetilde{U}\big(\overline{\widetilde{V}}_y-\widetilde{V}_y\big), \tag{16}$$

$$\widetilde{V}_x-(\widetilde{U}^2)_x=-i\big(\widetilde{V}_y+(\widetilde{U}^2)_y\big). \tag{17}$$

Now, in accordance with $\psi\to\tilde{\psi}$ following surfaces are constructed by $S\to S_t$ [6]:

$$S(x,y)=\begin{pmatrix} ix & -ie^{iy}chx \\ -ie^{-iy}chx & -ix \end{pmatrix}, \tag{18}$$

where the initial points on $u_0^1=u_0^3=0,u_0^2=1$,

$$S_t(x,y,t)=\begin{pmatrix} iu^3 & -u^1-iu^2 \\ u^1-iu^2 & -iu^3 \end{pmatrix}-i\int_0^t\begin{pmatrix} l & \bar{k} \\ k & -l \end{pmatrix}d\tau, \tag{19}$$

where

$$k(z,\bar{z},t)=\psi_{1,z}^2-\psi_{2,\bar{z}}^2-2\big(\psi_1\psi_{1,zz}-\psi_2\psi_{2,\overline{zz}}\big),$$

$$l(z,\bar{z},t)=\psi_{1,z}\overline{\psi}_{2,z}+\overline{\psi}_{1,\bar{z}}\psi_{2,\bar{z}}-\psi_{1,zz}\overline{\psi}_2-\psi_1\overline{\psi}_{2,zz}-\overline{\psi}_{1,\overline{zz}}\psi_2-\overline{\psi}_1\psi_{2,\overline{zz}},$$

will give some deformation of surface S.

For the inverted catenoid S_t corresponds one of nontrivial solutions $\widetilde{U}$ of the mVN equations (16), (17).

2. Inverted catenoid. If the surface $\psi:\mathcal{U}\to\mathbb{R}^3$ is minimal (for example, catenoid), then its inversion $\tilde{\psi}=T\cdot\psi$. Accordingly, the inversion of surface S (which passes through points $u_0=(0,1,0)$ with zero potential) is the following mapping:

$$S^{-1}:x\to-\frac{x}{|x|^2}$$

which transfer the catenoid to the surface S_t at some time $t=const$ at a point $x=0,y=0$ with potential $\widetilde{U}$.

In the following examples, for given minimal surfaces, their inversions are constructed by the Weierstrass representations (for surfaces $\tilde{\psi}$),the Gauss map (6), and the height differential (7).

Example 1. (Enneper surface) $\psi_1 = z, \psi_2 = 1$,

$$\tilde{\psi} = \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

By Weierstrass representations, we find the following components of this surface:

$$u^1(z,\bar{z}) = \int_{(0,0)}^{(z,\bar{z})} (z^2+1)dz - (\bar{z}^2+1)d\bar{z} = \frac{y^3}{3} - x^2y - y,$$

$$u^2(z,\bar{z}) = \int_{(0,0)}^{(z,\bar{z})} (1-z^2)dz + (1-\bar{z}^2)d\bar{z} - C = x - \frac{x^3}{3} + xy^2 - C,$$

$$u^3(z,\bar{z}) = \int_{(0,0)}^{(z,\bar{z})} zdz + \bar{z}d\bar{z} = x^2 - y^2,$$

here linear integrals do not depend on the integration path in the domain D, $C > 0$ *-some constant.*

Gauss map

$$G(z) = (z^2+1, 1-z^2, z)$$

depends on the choice of the initial point of the surface.

Stereographic projection is the mapping of a single sphere into a complex plane. In this example, the line that intersects the pole of the unit sphere and any other point of this sphere will be parallel to the complex plane, since the inverted Enneper surface $\tilde{\psi}$ translates the point $x = y = 0, t = C$ to ∞. In [6] were found blowing-up solutions of the mVN equation by rigid translation of the second coordinate axis of the initial Enneper surface ψ. Obviously, $dh = dz$ means there is no surface rotation.

Example 2.(catenoid) $\psi_1 = \frac{1}{\sqrt{2}} e^{-\frac{z}{2}}, \psi_2 = \frac{1}{\sqrt{2}} e^{\frac{\bar{z}}{2}}$.

$$u^1(z,\bar{z}) = \frac{i}{2}\int_{(0,0)}^{(z,\bar{z})} \left(\frac{e^z - e^{-z}}{2}dz - \frac{e^{\bar{z}} - e^{-\bar{z}}}{2}d\bar{z}\right) = -\text{ch}x \sin y,$$

$$u^2(z,\bar{z}) = \frac{1}{2}\int_{(0,0)}^{(z,\bar{z})} \left(\frac{e^z + e^{-z}}{2}dz + \frac{e^{\bar{z}} + e^{-\bar{z}}}{2}d\bar{z}\right) = \text{ch}x \cos y,$$

$$u^3(z,\bar{z}) = \int_{(0,0)}^{(z,\bar{z})} \left(\frac{1}{2}dz + \frac{1}{2}d\bar{z}\right) = x.$$

Inverted catenoid as solutions of equations (12)

$$\tilde{\psi}: \tilde{\psi}_1(z,\bar{z},t) = \frac{e^{-\frac{z}{2}-\frac{t}{8}}}{\sqrt{2}}, \tilde{\psi}_2(z,\bar{z},t) = \frac{e^{\frac{\bar{z}}{2}-\frac{t}{8}}}{\sqrt{2}}$$

also determined by Weierstrass representations

$$u^1(z,\bar{z},t) = -ch\left(x + \frac{t}{4}\right) \cdot \sin y,$$

$$u^2(z,\bar{z},t) = ch\left(x + \frac{t}{4}\right) \cdot \cos y - ch\frac{t}{4} + 1,$$

$$u^3(z,\bar{z},t) = x.$$

Gauss map

$$G(z) = \left(sh\, z, \text{ch z}, \frac{1}{2}\right),$$

depends on the choice of the initial point of the surface, and

$$dh = 0.$$

The stereographic projection maps each point of the unit sphere to the all point of the complex plane. It means that solutions of the mVN equation $\tilde{U}(x,y,t)$ are determined for all constants $t = const$, by smooth translation of the initial catenoid ψ along the second coordinate axis $u^2 = u^2 \pm t$.

3. Soliton deformation of inverted catenoid. By substituting in (19), deformation part of surface S is found by the following time dynamics:

$$k = \frac{1}{4}e^{-iy} sh\left(x + \frac{t}{4}\right), l = -\frac{3}{4}.$$

Surface

$$S_t=\begin{pmatrix} i(x+\frac{3t}{4}) & -ie^{iy}ch(x+\frac{t}{4}) \\ -ie^{-iy}ch(x+\frac{t}{4}) & -i(x+\frac{3t}{4}) \end{pmatrix}$$

is constructed by is constructed by S at some time $t=const$:

$$u^1(x,y)=-ch(x+\frac{const}{4})\mathrm{sin}y,$$
$$u^2(x,y)=ch\left(x+\frac{const}{4}\right)\mathrm{cos}y, \tag{20}$$
$$u^3(x,y)=x+\frac{const}{4},$$

with initial point $u_0^1=0, u_0^2=ch\frac{const}{4}, u_0^3=\frac{const}{4}$

Soliton deformation at the surface $S(x,y)=\begin{pmatrix} iu^3 & -u^1-iu^2 \\ u^1-iu^2 & -iu^3 \end{pmatrix}$, is determined by formulas (19).

In [6], it was shown that the following surface:

$$S_t(x,y,t)=S(x,y)-i\int_0^t\begin{pmatrix} l & \overline{k} \\ k & -l \end{pmatrix}d\tau$$

will give soliton deformation of the surface S by the Moutard transformations [5],[6].

We present the algorithm of Moutard transformations for surface S_t, obtained in [5],[6].

By this algorithm, we find W, A, B, C by introducing (see [6]) the following notation:

$$K(\Psi_0)=\begin{pmatrix} iW & A \\ -\overline{A} & -iW \end{pmatrix}, M(\Psi_0)=\begin{pmatrix} B & C \\ -\overline{C} & \overline{B} \end{pmatrix}.$$

$$\widetilde{U}=W, \tag{21}$$

$$\widetilde{V}=A^2+2\left(A\overline{B}-i\overline{C}W\right). \tag{22}$$

$$\boldsymbol{W=\frac{(const+1)cosy+(x+\frac{3t}{4})sh(x+\frac{t}{4})-ch(x+\frac{t}{4})}{(const+1)^2-2(const+1)cosy\cdot ch(x+\frac{t}{4})+(x+\frac{3t}{4})^2+ch^2(x+\frac{t}{4})}},$$

$$\boldsymbol{A=i\cdot\frac{(const+1)\cdot(cosy\cdot sh(x+\frac{t}{4})+isiny\cdot ch(x+\frac{t}{4}))-sh(x+\frac{t}{4})ch(x+\frac{t}{4})-x-\frac{3t}{4})}{(const+1)^2-2(const+1)cosy\cdot ch(x+\frac{t}{4})+(x+\frac{3t}{4})^2+ch^2(x+\frac{t}{4})}},$$

$$\boldsymbol{B=-\frac{i}{2}\cdot th(x+\frac{t}{4}), C=\frac{i}{2\cdot ch(x+\frac{t}{4})}}.$$

Then, by formulas (21), (22), we finally obtain solutions of the mVN equations (16, 17) for the inverted catenoid $u^2\rightarrow u^2-const$

$$\boldsymbol{\widetilde{U}(z,\bar{z},t)=}$$

$$\boldsymbol{=\frac{(const+1)(\frac{z-\bar{z}}{2})+(\frac{z+\bar{z}}{2}+\frac{3t}{4})sh(\frac{z+\bar{z}}{2}+\frac{t}{4})-ch(\frac{z+\bar{z}}{2}+\frac{t}{4})}{(const+1)^2-2(const+1)(\frac{z+\bar{z}}{2}+\frac{t}{4})\cdot ch(\frac{z-\bar{z}}{2})+(\frac{z+\bar{z}}{2}+\frac{3t}{4})^2+ch^2(\frac{z+\bar{z}}{2}+\frac{t}{4})}}$$

or

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) =$$

$$= \frac{(\boldsymbol{const}+\mathbf{1})\boldsymbol{cosy} + (\boldsymbol{x}+\frac{3t}{4})\boldsymbol{sh}(\boldsymbol{x}+\frac{t}{4}) - \boldsymbol{ch}(\boldsymbol{x}+\frac{t}{4})}{(\boldsymbol{const}+\mathbf{1})^{2} - 2(\boldsymbol{const}+\mathbf{1})\boldsymbol{cosy}\cdot \boldsymbol{ch}(\boldsymbol{x}+\frac{t}{4}) + (\boldsymbol{x}+\frac{3t}{4})^{2} + \boldsymbol{ch}^{2}(\boldsymbol{x}+\frac{t}{4})}, \quad (\mathbf{23})$$

$$\widetilde{\boldsymbol{V}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) = \frac{\boldsymbol{th}^{2}(\boldsymbol{x}+\frac{t}{4})}{\mathbf{4}} -$$

$$-\left(\frac{\boldsymbol{th}(\boldsymbol{x}+\frac{t}{4})}{\mathbf{2}}\right.$$

$$\left.++\frac{(\boldsymbol{const}+\mathbf{1})(\boldsymbol{cosysh}(\boldsymbol{x}+\frac{t}{4}) + \boldsymbol{isinych}(\boldsymbol{x}+\frac{t}{4})) - \boldsymbol{sh}(\boldsymbol{x}+\frac{t}{4})\boldsymbol{ch}(\boldsymbol{x}+\frac{t}{4}) - \boldsymbol{x} - \frac{3t}{4})}{(\boldsymbol{const}+\mathbf{1})^{2} - 2(\boldsymbol{const}+\mathbf{1})\boldsymbol{cosych}(\boldsymbol{x}+\frac{t}{4}) + (\boldsymbol{x}+\frac{3t}{4})^{2} + \boldsymbol{ch}^{2}(\boldsymbol{x}+\frac{t}{4})}\right)^{2} -$$

$$-\frac{\widetilde{\boldsymbol{U}}}{\boldsymbol{ch}\left(\boldsymbol{x}+\frac{t}{4}\right)}. \quad (\mathbf{24})$$

In particular, with condition $x = y = 0, t = 0$ new potential $\widetilde{U} = 0$. So we obtain

$$\widetilde{U}(0,0,t) = \begin{cases} \frac{1}{const}, if\ t > 0, u^2 \to u^2 - t, \\ 0, if\ t = 0, u^2 \to u^2, \\ -\frac{1}{const}, if\ t < 0, u^2 \to u^2 + t. \end{cases}$$

Therefore, deformation of the catenoid generates a smooth function $\frac{1}{const} sgn\ t$ at all points t except the zero of potential $\widetilde{U}$ at the point $x = y = 0$.

It is known that the derivative of the signum is equal to the Dirac delta function.

4. Main result.

Theorem.

1. The soliton deformation is determined by the smooth translation of the catenoid ψ along the second coordinate axis $u^2 = u^2 + 1$, and exact solution $\widetilde{U}_1(x,t)$ of following modified Karteweg-de Vries equation (mKdV) [15]: $\widetilde{U}_{1t} = \frac{1}{4}\widetilde{U}_{1xxx} + 6\widetilde{U}_{1x}\widetilde{U}_1{}^2$,
is found

$$\widetilde{\boldsymbol{U}}_{\mathbf{1}}(\boldsymbol{x},\boldsymbol{t}) = \frac{\left(\boldsymbol{x}+\frac{3t}{4}\right)\boldsymbol{sh}\left(\boldsymbol{x}+\frac{t}{4}\right) - \boldsymbol{ch}\left(\boldsymbol{x}+\frac{t}{4}\right)}{(\boldsymbol{x}+\frac{3t}{4})^{2} + \boldsymbol{ch}^{2}(\boldsymbol{x}+\frac{t}{4})}. \quad (\mathbf{25})$$

2. Inverted catenoid generates a smooth function $\frac{1}{\mathrm{const}}$ sgn t at all points t except the zero of potential $\widetilde{\mathrm{U}}$ at the point x = y = 0 and

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\widetilde{\boldsymbol{U}}(\mathbf{0},\mathbf{0},\boldsymbol{t}) = \frac{\mathbf{1}}{\boldsymbol{const}}\boldsymbol{\delta}(\boldsymbol{t})$$

where $\delta(t)$ is the Dirac delta function, $const \neq 0$ - nonzero constant.

The smooth translation of the catenoid ψ is also determined along the second coordinate axis $u^2 = u^2 \pm const$ until $const \neq 0$ and $\widetilde{U}(x,y,t)$, $\widetilde{V}(x,y,t)$ satisfy the mVN equations represented by (23), (24).

Proof. mKdV solution $\widetilde{U}_1(x,t)$ is obtained by simple substituting $const = -1$ in potential (23). Therefore potential $\widetilde{U}_1$ depends on variable x. By substituting $const = -1$ in potential (24), we obtain

$\tilde{V}_1(x,t)$, which satisfy $\tilde{V}_1 = \tilde{U}_1{}^2$. This implies the well-known fact that mVN equations can be reduced to mKdV equation. Note that potential representation of inverted catenoid satisfies mVN equations by Moutard transformations and inverted catenoid satisfies Airy type equations. Analogically potential representation (25) of mKdV surface satisfy mKdV equation by condition $const = -1$ and catenoid is intended to initial data (11). Second part of theorem is clearly.

The obtained results can also be applied in the physical sciences by considered as [12], [13], [16], [17].

Acknowledgments. This work was supported by the Ministry of Education and Science of Kazakhstan under grants AP08856912.

Д. Құрманбаев[1, 2], Қ. Есмаханова[3]

[1]Әл-Фараби атындағы Қазақ Ұлттық Университеті, Алматы, Қазақстан;
[2]Сүлеймен Демирел атындағы Университет, Қаскелең, Қазақстан;
[3]Л.Н. Гумилев атындағы Еуразия Ұлттық Университеті, Нұр-Сұлтан, Қазақстан;

ИНВЕРСИЯЛАНҒАН КАТЕНОИД ҮШІН СОЛИТОНДЫҚ ДЕФОРМАЦИЯ

Аннотация. Минималды бет ([1] қараңыз) үш өлшемді кеңістікте Вейерштрасс көрінісі арқылы анықталады. Спинор терминінде Дирак теңдеуінің ([2] жұмысындағы) шешімі изотермалды координаталары сақталған осы минималды бет арқылы ұсынылады. Манаковтың L,A,B үштігіне енетін Дирак операторы ([3] енгізілген) арқылы жазылатын теңдеу модификацияланған Веселов-Новиковтың теңдеуіне (мВН) ([4] қараңыз) эквивалентті болады. Дирак операторының U потенциалы минималды бетті ұсынатын потенциал болып табылады. Дирак операторының белгілі потенциалдары арқылы мВН теңдеуінің жаңа шешімдері құрастырылатын алгоритм Мутар түрлендіруі ([5]) деп аталады. Біріншіден, осы түрлендірудің [6], [7] жұмыстарында табылған геометриялық мағынасы минималды беттің инверсиясына анықтама береді, ары қарай мВН теңдеуінің нақты шешімдерін табу арқылы инверсияланған беттерді сипаттай аламыз. Бұл жаңа потенциалдардың сипаттамасы [8], [9] жұмыстарында енгізілген солитонды деформацияны анықтайды. 2014 жылы бастапқы Эннепер бетін қатаң жылжыту арқылы мВН теңдеуінің бұзушы шешімдері [6] жұмысында табылған. Ары қарай екінші ретті Эннепер беті үшін [10] жұмысында нәтижелер алынған. Енді екінші коор-динаталық осьтің бойымен тегіс жылжыту арқылы инверсияланған катеноид үшін солитондық деформация алынады.

Бұл жұмыста катеноидтың инверсиясын анықтау үшін Гаусс бейнелеуі, биік дифференциал ([11] қараңыз) деген голоморфты объектілерді табу ұсынылады; сонымен қатар, инверсияланған катеноидтың солитонды деформациясы алынды; модификацияланған Картевег-де-Вриз теңдеуінің (КдВ) дербес шешімі табылды, ал бұл өз кезегінде КдВ беттері туралы сипаттама береді([12],[13]).

Түйін сөздер: Модификацияланған Веселов-Новиков теңдеуі, Дирак операторы, Гаусс бейнелеуі, биік дифференциал, стереографикалық проекция, солитондық деформация, Мутар түрлендіруі, катеноид.

Д. Курманбаев[1, 2], К. Есмаханова[3]

[1]Казахский национальный университет им. аль-Фараби, Алматы, Казахстан;
[2]Университет им. Сулеймана Демиреля, Каскелен, Казахстан;
[3]Евразийский Национальный Университет им. Л.Н. Гумилева, Нур-Султан, Казахстан

СОЛИТОННАЯ ДЕФОРМАЦИЯ ИНВЕРСИРОВАННОГО КАТЕНОИДА

Аннотация. Минимальная поверхность (см.[1]) определяется с помощью представления Вейерштрасса в трехмерном пространстве. Решение уравнения Дирака [2] в терминах спиноров совпадает с представлениями этой поверхности с сохранением изотермических координат. Уравнение, представляемое через оператора Дирака, который входит в L,A,B тройку Манакова (см.[3]), равносильно модифицированному уравнению Веселова-Новикова (мВН) (см.[4]). Потенциал U оператора Дирака является потенциалом представления минимальной поверхности. Новые решения уравнения мВН строятся с помощью известных потенциалов оператора Дирака, и этот алгоритм называется преобразованием Мутара [5]. Геометрический смысл этого преобразования, найденный в работах [6], [7], во-первых, дает нам определение инверсии минимальной поверхности, далее, с нахождением точных решений уравнения мВН, мы можем представить инверсированные поверхности. А эти представления нового потенциала определяют солитонную деформа-

цию, введенную в работах [8] и [9]. В 2014 году были найдены разрушающие решения уравнения мВН с помощью жесткой трансляцией изначальной поверхности Эннепера в работе [6]. Дальнейшие результаты найдены в работе [10] при поверхности Эннепера второго порядка. Теперь находится солитонная деформация при инверсированного катеноида с помощью гладкой трансляцией второй координатной оси.

В данной работе для определения инверсий катеноида предлагается найти голоморфные объекты как отображения Гаусса и высокого дифференциала (см. [11]); также в работе получена солитонная деформация инверсированного катеноида; найдено частное решение модифицированного уравнения Картевега-де-Вриза (КдВ), что дает нам представление о КдВ-поверхностях (см.[12],[13]).

Ключевые слова: модифицированное уравнение Веселова-Новикова, оператор Дирака, отображение Гаусса, высокий дифференциал, стереографическая проекция, солитонная деформация, преобразование Мутара, катеноид.

Information about authors:

Kurmanbayev Damir Muratbekovich, PhD student at the Department of Fundamental Mathematics, Al-Farabi Kazakh National University, Almaty, Kazakhstan; senior-lecturer of the Department of Mathematics and Natural Science, Suleyman Demirel University, Kaskelen, Kazakhstan, kurmanbaev.damir@gmail.com, https://orcid.org/ 0000-0003-4824-8737;

Yesmakhanova Kuralay Ratbaevna, Associate Professor of the Department of Mathematical and Computer Modeling, L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan, kryesmakhanova@gmail.com, https://orcid.org/ 0000-0002-4305-5939

REFERENCES

[1] Taimanov I.A. (2008). Lectures on Differential Geometry. European Mathematical Society, 211 pages.

[2] Taimanov I.A.(2006). Two-dimensional Dirac operator and the theory of surfaces. Russian Math. Surveys 61, no. 1, 79-159.

[3] Manakov S.V. (1976). The method of the inverse scattering problem and two dimensional evolution equations. Uspekhi Mat. Nauk, Vol.31, 245-246.

[4] Bogdanov L.V. (1987). Veselov-Novikov equation as a natural two-dimensional generalization of the Korteweg-de Vries equation. Theor. Math. Phys. 70.

[5] Yu Delong, Liu Q.P., Wang Shikun. (2001). Darboux transformation for the modified Veselov Novikov equation. J. of Physics A35, 2001, 3779-3785.

[6] Taimanov I.A. (2015). Blowing up solutions of the modified Novikov-Veselov equation and minimal surfaces. Theoret. and Math. Phys., 182:2, 173-181.

[7] Taimanov I.A. (2015). The Moutard Transformation of Two-Dimensional Dirac Operators and MoЁbius Geometry. Math. Notes, 97:1, 124–135.

[8] Konopelchenko B.G. (1996) Induced surfaces and their integrable dynamics. Stud. Appl. Math., 96:1, 9-51.

[9] Taimanov I.A. (1997) Modified Novikov-Veselov equation and differential geometry of surfaces. Amer. Math. Soc. Transl., Ser. 2, V. 179, pp. 133-151.

[10] Kurmanbayev D. (2018) Construction blowing up solutions of modified Novikov-Veselov equation by second order Enneper surface. Journal of Mathematics, Mechanics and Computer Science, [S.l.], v. 84, n. 1, p. 77-85.

[11] Weber M. (2001). Classical Minimal Surfaces in Euclidean Space by Examples.Claynotes.

[12] Perel'man T. L., Fridman A.Kh., and El'yashevich M. M. (1974). A modified Korteweg-de Vries equation in electrohydrodynamics. Zh. Eksp. Teor. Fiz. Vol. 39, No.4, 1316-1323.

[13] Castro-Villarreal P., Guven J. (2007) Inverted catenoid as a fluid membrane with two points pulled together. Physical Review E 76, 011922.

[14] Kurmanbayev D. (2020). Exact Solution of Modified Veselov-Novikov Equation and Some Applications in the Game Theory. International Journal of Mathematics and Mathematical Sciences, ID 9740638, Vol.2020, 1-7.

[15] Kaya D. (2009). Korteweg-de Vries Equation (KdV) and Modified Korteweg-de Vries Equations (mKdV), Semi-analytical Methods for Solving the. Encyclopedia of Complexity and Systems Science. 5143-5161. doi:10.1007/978-0-387-30440-3, 305.

[16] Sym A. (1985). Soliton Surfaces and Their Applications. Lecture Notes in Physics. Vol. 239, p.154, Springer, Berlin.

[17] Yesmakhanova K., Bekova G., Myrzakulov R. (2017). Darboux transformation and soliton solution for the (2+1)-dimensional complex modified Korteweg-de Vries equations. Journal of Physics: Conference Series, 6th International Conference on Mathematical Modelling in Physical Sciences (IC-MSQUARE 2017), Paphos, Cyprus.